Nonvoting and the Existence of Equilibrium under Majority Rule*

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I. INTRODUCTION

Consider two political candidates who are confronted with the task of selecting electoral platforms, i.e., of espousing positions on one or more issues before an electorate. Suppose that each candidate wishes to select a platform which secures a majority of the votes cast, or which at least secures a tie. Assuming that citizens possess well-defined utility functions over the space of all admissible platforms, the candidates might be confronted with the dilemma implied by Arrow's Impossibility Theorem [1]. That is, an electoral equilibrium may not exist in that every admissible platform is defeated in a majority vote. Whether or not such an equilibrium exists depends on the alternatives to Arrow's conditions that are assumed. Black [2], Inada [7], and Sen [13] formulate constraints on individual preference orderings which are sufficient to establish transitive social orderings, in which case an electoral equilibrium clearly exists. Similarly, Davis, DeGroot and Hinich [4], Plott [10], Tullock [16], Sen and Pattanaik [14], and Simpson [15] formulate constraints on individual utility functions and on the distribution of citizens' ideal points which yield an equilibrium (but not necessarily a completely transitive social ordering). Nevertheless, the restrictiveness of the conditions considered thus far leads Kramer [8] to the assertion that "cyclical majorities, or the absence of stable outcomes under majority rule, are a fact of life." (p. 25)

Such pessimism, however, is unwarranted. With the exception of Sen's consideration of the utility and the disutility of voting (see [12]), this research is an elaboration of Black's analysis of single-peaked preference orderings. That is, these sufficient conditions consist of restrictions on

the form of individual utility functions in that they permit citizens to hold certain preferences but not others. But by considering the possibility that citizens fail to reveal their preferences (i.e., that they abstain from voting) we formulate in this essay an alternative type of sufficient condition. Specifically, we show in Theorem 1 that if citizens abstain in a prescribed manner, an electoral equilibrium exists for the general class of continuously differentiable concave utility functions. Thus, abstentions yield a majority choice for circumstances in which the assumption that every citizen votes fails to yield such a choice. We demonstrate in Theorem 2 that the imposition of a set of convex constraints on the domain of each candidate's platform does not invalidate Theorem 1. Finally, we briefly examine some special cases that establish the vector of mean preferences as the equilibrium.

II. THE MODEL AND MAIN RESULTS

Consider a society whose economy consists of a private commodity space Y and a public commodity space X. Let C be the index set for the citizens in the society, and let $w(y \mid c, x)$ be the utility of citizen $c \in C$ for $y \in Y$ if the public decision $x \in X$ is adopted. Let $A(c, x) \in Y$ be the set of admissible y which c can choose, given x, and let

$$u(c, x) = \max_{y \in A(c, x)} w(y \mid c, x).$$

Thus, $u(c, x)$ is the utility citizen c derives from the public decision x. It is assumed throught this paper that

(i) X is a compact convex subset of R^n.

(ii) For all $c \in C$, $u(c, x)$ is a continuously differentiable concave function of x, and is strictly concave for some subset of C of positive measure.

The distribution of preferences over the society for the elements of X is given by a probability measure defined on C as follows: The probability that a randomly chosen citizen prefers x_1 to x_2 or is indifferent between them is $\mu(S)$ where $S = \{c \mid u(c, x_1) \geqslant u(c, x_2)\}$, assuming that S is measurable.[1]

Suppose there are two candidates whose proposed platforms are θ and $\psi \in X$. Assume that the citizens have identical perceptions of these two

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[1] We do not require for our analysis that C be finite. However, our results still hold for the special case of a finite citizenry.

platforms, and believe them to be the social decisions which the candidates adopt if elected. Let $h^1(u(c, \theta), u(c, \psi), c)$ denote the probability that citizen c votes for θ given ψ, and assume that $h^1 = 0$, if $u(c, \theta) < u(c, \psi)$, and $h^1 = \gamma \geq 0$ if $u(c, \theta) = u(c, \psi)$, i.e., the citizen will vote for θ with probability γ if *he* is indifferent between θ and ψ. Similarly, let $h^2(u(c, \theta), u(c, \psi), c)$ denote the probability that citizen c votes for ψ given θ, (and also assume that $h^2 = 0$ if $u(c, \theta) > u(c, \psi)$, and $h^2 = \gamma \geq 0$ if $u(c, \theta) = u(c, \psi)$).

Abstention occurs, then, because $h^1 + h^2$, generally, is permitted to be less than 1. We require, moreover, that the formulation of abstention permitted in this analysis be consistent with the two causes of abstention identified in studies of voting participation: *alienation* and *indifference* (see Davis, Hinich, and Ordeshook [5]). Specifically, the alienation hypothesis states that, *ceteris paribus*, a citizen's probability of voting diminishes as the utility he associates with his most preferred candidate decreases. Assume that $u(c, \theta) \geq u(c, \psi)$ so that the citizen's probability of voting is h^1; we require then that h^1 decreases as $u(c, \theta)$ decreases. And the indifference hypotheses states that *ceteris paribus* a citizen's probability of voting diminishes as the utility differential between both candidates decreases. This requires that as either $u(c, \theta)$ approaches $u(c, \psi)$ or $u(c, \psi)$ approaches $u(c, \theta)$, h^1 diminishes.[2]

It follows, now, from the definitions of h^1 and h^2 that the probability that a citizen, randomly chosen from the society, votes for θ given ψ is

$$V^1(\theta, \psi) = \int h^1(u(c, \theta), u(c, \psi), c) \, d\mu(c), \qquad (1)$$

and the probability that a randomly chosen citizen votes for ψ given θ is

$$V^2(\theta, \psi) = \int h^2(u(c, \theta), u(c, \psi), c) \, d\mu(c). \qquad (2)$$

For a single election the candidate who wins is the one who receives a positive plurality of the votes cast. In this paper, the political process is

[2] Observe that we let u, h^1, and h^2 be functions of c, the citizen index. That is, we do not assume that the functional forms of u, h^1, or h^2 be identical for any subset of citizens, or even that these functional forms bear any correspondence across citizens. Consider, for example, $u(c, \theta)$: Davis, DeGroot, and Hinich [4] assume that, for all $c \in C$, $u(c, \theta)$ is a monotonically decreasing function of the quadratic metric $(\theta - x(c))' A (\theta - x(c))$, where A is identical for all $c \in C$. Similarly, Plott [10] requires that all contract loci of all pairs of citizens who prefer diametrically opposed positions pass through a common point (pp. 790–791). Our analysis, however, imposes neither the Davis *et al.* nor the Plott restrictions.

modeled as a two person noncooperative game where φ^1 and φ^2 are the respective objective functions of the two candidates:

$$\varphi^1(\theta, \psi) = V^1(\theta, \psi) - V^2(\theta, \psi), \qquad (3)$$

$$\varphi^2(\theta, \psi) = V^2(\theta, \psi) - V^1(\theta, \psi). \qquad (4)$$

Assume that each candidate adjusts his platform continuously as follows:

$$d\theta_i/dt = r^1 \, \partial\varphi^1/\partial\theta_i, \qquad i = 1,...,n, \quad r^1 > 0, \qquad (5a)$$

$$d\psi_j/dt = r^2 \, \partial\varphi^2/\partial\theta_j, \qquad j = 1,...,n, \quad r^2 > 0. \qquad (5b)$$

THEOREM 1. 1. *For each* $c \in C$ *and for* $k = 1, 2, h^k(q_1, q_2, c)$ *is continuously differentiable and*

$$\lim_{q_1 \to q_2^+} h_j{}^1 = \lim_{q_2 \to q_1^-} h_j{}^2.$$

2. *Furthermore,* $h_1{}^1 \geq 0$, $h_1{}^2 \leq 0$, $h_{11}^1 \leq 0$, $h_{22}^1 \geq 0$, *and* $h_1{}^2 \leq 0$, $h_2{}^2 \geq 0$, $h_{11}^2 \geq 0$, $h_{22}^2 \leq 0$, *where the inequalities are strict for those* c *for which* $u(c, x)$ *is strictly concave. Then,*

A. There exists a unique equilibrium for the zero-sum noncooperative political game, (α, α), *where*[3]

$$\varphi^1(\alpha, \psi) \geq \varphi^1(\alpha, \alpha) \qquad \text{for all } \psi$$

and

$$\varphi^2(\theta, \alpha) \geq \varphi^2(\alpha, \alpha) \qquad \text{for all } \theta.$$

B. From any initial (θ^0, ψ^0), *the solution to* (5), *call it* $(\theta(t), \psi(t))$, *exists and converges to* (α, α), *i.e.,* $\lim_{t \to \infty} \theta(t) = \alpha$ *and* $\lim_{t \to \infty} \psi(t) = \alpha$.

Proof. These results follow from Theorems 1, 2, 7, 8, and 9 of Rosen [11]. To apply these theorems we must show that the matrix

$$Q(\theta, \psi) = \begin{pmatrix} 2\nabla_{\theta\theta}\varphi^1 & \nabla_{\theta\psi}\varphi^1 + \nabla_{\theta\psi}\varphi^2 \\ \nabla_{\psi\theta}\varphi^1 + \nabla_{\psi\theta}\varphi^2 & 2\nabla_{\psi\psi}\varphi^2 \end{pmatrix} \qquad (6)$$

is negative definite, where $\nabla_{\theta\theta}\varphi^k$, $\nabla_{\theta\psi}\varphi^k$, $\nabla_{\psi\theta}\varphi^k$, $\nabla_{\psi\psi}\varphi^k$ are the Jacobian matrices $(\partial^2\varphi^k/\partial\theta_i \, \partial\theta_j)$, $(\partial^2\varphi^k/\partial\theta_i \, \partial\psi_j)$, $(\partial^2\varphi^k/\partial\psi_i \, \partial\theta_j)$, and $(\partial^2\varphi^k/\partial\psi_i \, \partial\psi_j)$,

[3] Thus if one candidate chooses α as his platform and the other candidate chooses another platform, the candidate choosing α gains a positive plurality and wins.

respectively, for $k = 1, 2$. Note that since $\varphi^1 = -\varphi^2$, Q can be written

$$Q = 2 \begin{pmatrix} \nabla_{\theta\theta}\varphi^1 & 0 \\ 0 & -\nabla_{\psi\psi}\varphi^1 \end{pmatrix}. \tag{7}$$

Let $f(q_1, q_2, c) = h^1(q_1, q_2, c) - h^2(q_1, q_2, c)$. From (1), (2), and (3),

$$\varphi^1(\theta, \psi) = \int f(u(c, \theta), u(c, \psi), c) \, d\mu(c) \tag{8}$$

Moreover, $f(q_1, q_1, c) = 0$ for $q_1 = q_2$ since $h^k = \gamma$, and

$$\begin{aligned} f(q_1, q_2, c) &= h^1(q_1, q_2, c) & \text{for } q_1 > q_2 \\ &= -h^2(q_1, q_2, c) & \text{for } q_1 < q_2. \end{aligned} \tag{9}$$

Thus $f_1 \geq 0, f_2 \leq 0, f_{11} \leq 0$, and $f_{22} \geq 0$ for all $q_1, q_2 \in R$ and $c \in C$ with the inequalities strict for c in some subset of positive measure.

Let $\nabla_\theta u(c, \theta) = (\partial u/\partial \theta_1, ..., \partial u/\partial \theta_n)$ and $\nabla_{\theta\theta} u(c, \theta)$ be the matrix $(\partial^2 u/\partial \theta_i \, \partial \theta_j)$, which is negative semidefinite by the concavity of u and is negative definite for c in the subset of positive measure where u is strictly concave. Thus

$$\nabla_{\theta\theta} f = f_{11}(\nabla_\theta u)(\nabla_\theta u)' + f_1 \nabla_{\theta\theta} u$$

is negative semidefinite for all q_1, q_2, c, and negative definite for c in a subset of positive measure.

Thus by integrating with respect to $\mu(c)$, the matrix

$$\nabla_{\theta\theta}\varphi^1 = \int (\nabla_{\theta\theta} f) \, d\mu(c)$$

is *negative definite*. In a similar way, it can be shown that $\nabla_{\psi\psi}\varphi^1$ is *positive definite* and thus from (7), $Q(\theta, \psi)$ is negative definite. Moreover, the objective function φ^1 is a strictly concave function of θ and the objective function φ^2 is a strictly concave function of ψ.

It thus follows from Theorems 1, 2, 7, 8, 9 of Rosen [11] that there exists a unique equilibrium (θ^*, ψ^*) such that $\lim_{t\to\infty} \theta(t) = \theta^*$ and $\lim_{t\to\infty} \psi(t) = \psi^*$.[4]

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[4] Strictly speaking, Theorem 8 does not directly apply. However, if $\| (\theta/r^1, \psi/r^2) \|$ is used as a Lyapunov function in place of the $\| (\theta, \psi) \|$ used in his paper, then his proof is adapted to our case.

To prove that $\theta^* = \psi^* = \alpha$, suppose that $\theta^* \neq \psi^*$. Since $\varphi^1(\theta^*, \psi^*) \geq \varphi^1(\psi^*, \psi^*) = 0$ and $\varphi^2(\theta^*, \psi^*) \geq \varphi^2(\theta^*, \theta^*) = 0$, it follows from $\varphi^1 = -\varphi^2$ that

$$\varphi^1(\theta^*, \psi^*) = \varphi^2(\theta^*, \psi^*) = 0.$$

Hence, by strict concavity of φ^1 in θ,

$$\varphi^1(\lambda\theta^* + (1 - \lambda)\psi^*, \psi^*) > \lambda\varphi^1(\theta^*, \psi^*) + (1 - \lambda)\varphi^1(\psi^*, \psi^*)$$
$$= 0 = \varphi^1(\theta^*, \psi^*).$$

Hence, (θ^*, ψ^*) is not an equilibrium, unless $\theta^* = \psi^*$.

By the strict concavity of the objective functions, the zero-sum nature of the game, and the nature of the equilibrium, it follows that $\varphi^1(\alpha, \psi) \geq 0$ and $\varphi^2(\theta, \alpha) \geq 0$. The social decision α is dominant.   Q.E.D.

Let us assume now that each candidate's choice of a platform is constrained to be in the set defined by

$$G^j(\theta, \psi) \geq 0, \qquad \text{for } j = 1, ..., J,$$

given J constraint functions $G^1, ..., G^J$. In this case let each candidate adjust its platform as follows:

$$\begin{aligned} \frac{d\theta_i}{dt} &= r^1 \frac{\partial\varphi^1}{\partial\theta_i} + \sum_{j=1}^J \delta_j \frac{\partial G^j}{\partial\theta_i} \\ &= a_i(\theta, \psi, \delta) \qquad \text{for } i = 1, ..., n, \end{aligned} \tag{10a}$$

$$\begin{aligned} \frac{d\psi_i}{\partial t} &= r^2 \frac{\partial\varphi^2}{\partial\psi_i} + \sum_{j=1}^J \lambda_j \frac{\partial G^j}{\partial\psi_i} \\ &= b_i(\theta, \psi, \lambda) \qquad \text{for } i = 1, ..., n, \end{aligned} \tag{10b}$$

where δ is chosen to minimize $\| a \| = [\sum_{i=1}^n a_i^2]^{1/2}$ subject to

$$\begin{aligned} \delta_j &\geq 0, \qquad \text{if } G^j = 0, \\ \delta_j &= 0, \qquad \text{if } G^j > 0, \end{aligned}$$

and similarly for λ. Thus $d\theta/dt$ is the projection of the gradient of $r^1\varphi^1$ onto the binding constraint manifold.

COROLLARY 1.1. *Suppose that conditions 1 and 2 of Theorem 1 hold.*

If G^j is concave for $j = 1,..., J$ and if there is no j such that both $\partial G^j/\partial \theta_i \neq 0$ and $\partial G^j/\partial \psi_i \neq 0$ for some i,[5] then the conclusions of Theorem 1 hold for the adjustment Eqs. (10).

When the possibility of coupled constraints is introduced (i.e., some j, i, k such that $\partial G^j/\partial \theta_i \neq 0$ and $\partial G^j/\partial \psi_k \neq 0$ hold), the results become less general. Although an equilibrium combination of platforms still exists, uniqueness is not guaranteed. In addition, the equilibrium may depend on the speeds of adjustment (r^1, r^2). (Rosen calls these normalized equilibria.) We can, however, state the following for $r^1 = r^2$.

THEOREM 2. *Suppose that conditions 1 and 2 of Theorem 1 hold. If $r^1 = r^2 > 0$ and if G^j is concave for all $j = 1,..., J$, then for any initial (θ^0, ψ^0) the solution to (10) exists, call it $(\theta(t), \psi(t))$, and converges to a unique equilibrium (θ^*, ψ^*). That is, $\lim_{t\to\infty} \theta(t) = \theta^*$ and $\lim_{t\to\infty} \psi(t) = \psi^*$.*

COROLLARY 2.1. *If $G^j(x, x) \geqslant 0$ for all $x \in X$ and all $j = 1,..., J$, then $\theta^* = \psi^* = \alpha$.*

Proof. This follows from Theorems 4, 7, and 9 in Rosen [11] in much the same way as Theorem 1 does.

In order to prove Corollary 2.1, assume that $\theta^* \neq \psi^*$. Since (ψ^*, ψ^*) is feasible, it can be compared with (θ^*, ψ^*). Applying the appropriate part of the proof of Theorem 1, it follows that (θ^*, ψ^*) is not a equilibrium.

III. SOME SPECIAL CASES

Let us return to the situation where there are no constraints on θ and ψ other than that they belong to X. For some more restricted classes of measures μ, preferences $u(c, x)$, and abstention function $h^k(q_1, q_2, c)$, it is possible to ascertain some properties of the equilibrium, or dominant, platform α. In particular, three results obtain. If preferences and abstentions are radially symmetric about θ°, then $\alpha = \theta^\circ$. For certain abstention functions, α is the mean ideal preference. If, in addition, utility functions satisfy some additional conditions, then α maximizes the sum of utilities for all positive linear transformations of these utilities. These results are now stated without proof since they follow from the theorems of Section II.

[5] That is, the actions of candidate 1 are constrained independently of the actions of candidate 2, and vice versa.

RESULT 1. *Assume there exists $\theta^\circ \in X$ and a one-to-one mapping $s : C \to C$ such that*

(a) *for all $\theta \in X$, $c \in C$, $u(c, \theta^\circ + \theta) = u(s(c), \theta^\circ - \theta)$*

(b) *for all $A \subseteq C$, $\mu(A) = \mu[s(A)]$, and*

(c) *for all q_1, q_2, all $c \in C$ and $k = 1, 2$,*

$$h^k(q_1, q_2, c) = h^k(q_1, q_2, s(c)).$$

Assume also that the conditions of Theorems 1 hold. Then, $\alpha = \theta^\circ$ is the equilibrium social decision.

RESULT 1a. *If conditions a, b, c above hold and if the conditions of Corollary 2.1 hold, then $\alpha = \theta^\circ$.*

It would be useful to illustrate Result 1 by two examples. Suppose that C is finite and μ is the counting measure, i.e.,

$$V^1(\theta, \psi) = \sum_{c \in C} h(u(c, \theta), u(c, \psi), c)/n$$

(where n is the number of citizens) or, one man-one vote. Suppose that there are an even number of citizens in C and that for each c^+ there is an opposite c^- such that $u(c^+, \theta) = u(c^-, -\theta)$ for all $\theta \in X$. Assume that $h(q_1, q_2, c) = h(q_1, q_2, s(c))$. Then Result 1 holds and $\alpha = 0$.

Consider the special case where $u(c, \theta)$ is a monotonically decreasing function of $\| \theta - x(c)\|$, the Euclidean distance between $x(c) \in X$ and the point θ. The social decision $x(c)$, called the ideal or bliss point, is the one in X which gives the citizen c the maximum utility. If the distribution of the $x(c)$ is radially symmetric about zero, then the society is composed of pairs c^+, c^- such that $u(c^+, \theta) = u(c^-, \theta)$, and thus $\alpha = 0$ is the dominant social decision.

Observe that this last example is consistent with the sufficient conditions for the existence of a majority preference established by, e.g., Plott [10] and Simpson [16]. Nevertheless, with suitable restrictions on h^1, h^2, and μ, we can show that the mean ideal preference is the dominant social choice for conditions which fail to satisfy those of Plott and Simpson. Specifically, we do not impose now any constraints of symmetry on the distribution of $x(c)$. Suppose that for each c, $u(c, \theta)$ is *strictly concave* in $\theta \in X$. Since X is a compact set, there exists a unique point $x(c) \in X$, called the ideal or bliss position for c, which maximizes his utility, i.e.,

$$u(c, x(c)) = \max_{\theta \in X} u(c, \theta).$$

Assume that $x(c)$ is Borel measurable and let $F(x)$ be the distribution function induced by the mapping $x : C \to X$.

RESULT 2. *Assume that $x(c) = x(c')$ implies that $u(c, \theta) = u(c', \theta)$ for all $\theta \in X$ and that $f(q_1, q_2, c) = f(q_1, q_2, c')$ for all q_1, q_2. [Then $\varphi^1(\theta, \psi) = \int_X f(u(x, \theta), u(x, \psi), x) \, dF(x)$ is well defined.] Assume also, that for $i = 1,\ldots, n$ and all $\theta = \psi$, and $k = 1, 2$, $\partial h^k / \partial \theta_i$ is linear in x. Then*

$$\alpha = \int_X x \, dF(x).$$

RESULT 3. *If, in addition to the conditions of Result 2, $\partial u(x, \theta) / \partial \theta_i$ is linear in x for all θ, then*

$$\int_X u(x, \alpha) \, dF(x) = \max_\theta \int_X u(x, \theta) \, dF(x).$$

Moreover, for all linear transformations g such that $dg/du \geqslant 0$,

$$\int_X g[u(x, \alpha)] \, dF(x) = \max_\theta \int_X g[u(x, \theta)] \, dF(x)$$

Thus if h^1, h^2, and u have derivatives with respect to θ_i which are linear in the ideal position x, the mean ideal position of the society is the dominant social choice arrived at by two-party competition, and this mean ideal position maximizes the averages utility for the society.

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